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APPLICATIONS OF SUBSTITUTION GROUPS.

By G. A. MILLER, Ph. D., Paris, France.

Lagrange seems to have been the first to give a clear statement and at least a partial proof* of the following fundamental

THEOREM I. *The number* (N) *of different formal values which are obtained by permuting the n elements of a given function in every possible manner is a divisor of $n!$.*†

About thirty years later Ruffini proved that N cannot have the values 3 or 4 when $n=5$, in his work, "*Teoria generale delle equazioni, in cui si dimostra impossibile la soluzione algebraica delle equazioni generali di grado superiore al quarto*," Bologna, 1799. He thus proved also that N cannot be equal to every divisor of $n!$.

As it was known that the value of N is the quotient obtained by dividing $n!$ by the order of the largest substitution group which transforms the function into itself it became an important problem to determine all the possible orders of the substitution groups of n elements, especially since it was believed that this would throw light on the solution of the general equation of the n^{th} degree. This problem has been solved only for small values of n.

The given theorem of Lagrange indicates the most direct application of substitution groups and therefore naturally furnished the starting point for the early investigations in this subject. It may be readily proved in the following manner.‡

Let ψ, the given function, be unchanged only by the substitutions in the

*The proof given by Lagrange in his article, "*Reflexions sur la resolution algebrique des equations,*" Memoires de l'Academie de Berlin, 1770 and 1771, seems to have been generally considered as complete. Cf. Mathieu, Comptes Rendus, 46, page 1047. Burkhardt, on the contrary, seems to regard it as incomplete. Cf. Zeitschrift fur Mathematik, 1892, page 141.

†$n!=1.2.3.\ldots.n$.

‡Cf. Netto's Theory of Substitutions (Cole's edition) §41.

first row of the following rectangle. (These form a group, for the product of any two leaves ψ unchanged and is therefore found in this row.)

$$\begin{array}{llll}
s_1=1 & s_2 & s_3 \ldots\ldots \quad \ldots\ldots s_\alpha & \\
t_2 & s_2 t_2 & s_3 t_2 \ldots\ldots\ldots\ldots s_\alpha\, t_2 & \\
t_3 & s_2 t_3 & s_3 t_3 \ldots\ldots\ldots\ldots s_\alpha\, t_3 & \\
\vdots & \vdots & \vdots \quad \ldots\ldots\ldots\ldots \quad \vdots & \\
\vdots & \vdots & \vdots \quad \ldots\ldots\ldots\ldots \quad \vdots & \\
\vdots & \vdots & \vdots \quad \ldots\ldots\ldots\ldots \quad \vdots & \\
t_m & s_2 t_m & s_3 t_m \ldots\ldots\ldots\ldots s_\alpha\, t_m & m=\dfrac{n!}{\alpha}
\end{array}$$

ψ will assume the same formal value if any one of the substitutions of a given row is applied to it, for the first factor leaves it unchanged and the second factor is the same throughout the row. If we assume that t_β ($\beta=2, 3, \ \ldots\ldots m$) is not found in a preceding row the substitutions of the rectangle are all different, for from

$$t_{\beta_2} s_{\gamma_2} = t_{\beta_1} s_{\gamma_1} \qquad (\beta_2 \overline{\gtrless} \beta_1 \overline{\lessgtr} m)$$

we would have $\qquad (\gamma_1, \gamma_2, \gamma_3 \overline{\lessgtr} \alpha)$

$$t_{\beta_2} = t_{\beta_1} s_{\gamma_3}.$$

This is impossible unless $\beta_1=\beta_2$ and $s_{\gamma_3}=1$. In this case $t_{\beta_2} s_{\gamma_2}$ and $t_{\beta_1} s_{\gamma_1}$ occupy the same place in the given rectangle.

Since there are just $n!$ substitutions of n elements the given rectangle contains each substitution once and only once. If t_{β_1} and t_{β_2} would transform ψ into the same function (ψ_1) then would the products of all the substitutions in the rows containing t_{β_1} and t_{β_2} into a substitution* t_γ which transforms ψ_1 into ψ give 2α different substitutions that transform ψ into itself. This is contrary to the hypothesis. Therefore $N=m=$ a divisor of $n!$.

One of the best known functions to which these elementary principles of substitution groups are commonly applied is the anharmonic ratio of four points.† If the four points are represented by A, B, C, and D, their anharmonic or cross ratio may be represented by

$$\psi \equiv \frac{AB}{CB} \div \frac{AD}{CD} \text{ or } \frac{AB.CD}{AD.CB}.$$

It is required to find the number of formal values of ψ when the points are interchanged in every possible manner. We may do this by dividing $4!=24$ by the order of the largest group of degree four that transforms ψ into itself. Since ψ is unchanged by the substitution $AB.CD$ and also by the substitution

*Since the rectangle contains all the possible substitutions of n elements, it must contain the inverse of each of its substitutions. We shall always consider n to be a finite number.

†Cf. Harkness and Morley's.

$AD.BC$, it must be unchanged by the group generated by these substitutions, viz., by $(AB.CD)_4$. We know that there are only three groups* of degree four which include $(AB.CD)_4$ and that these contain either a substitution of the form AB or one of the form ABC. As no such substitution transforms ψ into itself $(AB.CD)_4$ is the largest group that has this property. The number of different values of ψ is therefore $24 \div 4 = 6$.

To find these six values of ψ we may arrange the substitutions of four elements as follows:

1	$AB.CD$	$AC.BD$	$AD.BC$
AB	CD	$ACBD$	$ADBC$
AC	$ABCD$	BD	$ADCB$
AD	$ABDC$	$ACDB$	BC
ABC	ACD	BDC	ADB
ACB	BCD	ABD	ADC

Since all the substitutions of a row transform ψ into the same function we can find the six formal values of ψ by applying to it the six substitutions of the first column in this rectangle.† We thus obtain the following, in order:

$$\frac{AB.CD}{AD.CB}=k\,;\ \frac{BA.CD}{BD.CA}=\frac{\dfrac{BA.CD}{AD.CB}}{\dfrac{AD.CB-AB.CD}{AD.CB}}=\frac{k}{k-1}\,;\ \frac{CB.AD}{CD.AB}=\frac{1}{k}.$$

$$\frac{DB.CA}{DA.CB}=\frac{AD.CB-AB.CD}{AD.CB}=1-k\,;\ \frac{BC.AD}{BD.AC}=\frac{1}{1-k};\ \frac{CA.BD}{CD.BA}=\frac{k-1}{k}=1-\frac{1}{k}.$$

This example furnishes also a clear illustration of what we mean by "*different formal values.*" The six given values of ψ are all different as to form but k may have such values that they are not all really different. E. g., if $k=-1$, they coincide in pairs. In this case the ratio is called *harmonic.* If $k=$ an imaginary cube root of -1, they coincide in triplets and the ratio is called *equianharmonic.*

It should be observed that each one of the four subgroups of $(ABCD)$all, which are of the form (ABC)all, has one substitution in each row. Hence the following

THEOREM II. *The six different formal values of an anharmonic ratio of four points may be obtained by transforming any three of its points symmetrically.*

*It is evident that the function is not symmetric. It would therefore only be necessary to examine it with respect to the other two groups.

†This is clearly only one of the 4096 different ways in which the six transforming substitutions may be selected.

If G is the largest group which transforms a function (ψ) into itself we say that *ψ belongs to G.* The same relation is also expressed by saying that *G belongs to ψ.** The former of these two expressions is to be preferred since only *one group belongs to any given ψ while an infinite number of functions belong to any given G.* This may be readily proved as follows :†

We first suppose that G is the symmetric group of n elements. Every symmetric function of these elements will then belong to G. That their number is infinite, follows directly from the fact that both a and b can have an infinite number of different values without impairing the symmetry of the following functions :‡

$$x_1{}^a+x_2{}^a+x_4{}^a+\ldots\ldots+x_n{}^a+bx_1x_2x_3\ldots\ldots x_n \quad \ldots\ldots\ldots\ldots \quad A.$$

We may now suppose that G consists of a single substitution, viz., identity. In this case every function of the n elements which is changed in form by each substitution of these elements belongs to G. If we suppose that a_1, a_2, a_3, ..., a_n represent n different given numbers, the following function belongs to G:

$$a_1x_1+a_2x_2+a_3x_3+\ldots\ldots+a_nx_n \quad \ldots\ldots\ldots\ldots\ldots\ldots B.$$

We may now assign all the possible values of a_1 with the exception of the finite number of values represented by a_2, a_3, ..., a_n. In this way we obtain an infinite number of functions belonging to G.

We finally suppose that G represents any other group whose order is g. If we apply the substitutions of G to any one of the functions of B we obtain g different functions, ψ_1, ψ_2, ψ_3, ..., ψ_g. In any of the functions A we may suppose $n=g$ and the x's, in order, replaced by these ψ's. The resulting function belongs to this G. It is clear that we obtain an infinite number of such functions even by using a particular function of either A or B. We did not prove that all the functions belonging to G can be obtained in this way. In fact, this is not the case. As it follows from the definition that only one group belongs to a given function the proof is complete.

We have thus far only considered the relations between groups and functions when all the elements of the function which are permuted and no others are explicitly contained in the corresponding group. We have also only considered the number of values of a function when its elements are permuted according to the symmetric group. That the arguments which were employed apply to much more general cases may be illustrated by means of the following well-known Trigonometry formula

$$\text{Sin}\,\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}.$$

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*Cf. Netto, Theory of Substitutions (Cole's Edition) §28.

†Ibid, §30.

‡ If a and b are complex numbers, A represents ∞^4 different functions.

If we regard the first member of this equation as a function of the three angles A, B, C of a triangle it belongs to the group (BC) and is therefore a three-valued function of the angles. The second member belongs to the group (bc) and is therefore a three-valued function of the sides. Hence the formula says that *a given three-valued function of the angles is equal to a given three-valued function of the sides.*

As no special properties were imposed upon any the sides or angles in deriving the formula the three different values of the angles must correspond separately to the three different values of the sides. It remains only to find the substitutions which transform the given formula into the other two. To do this we may arrange the substitutions of the angles and the sides, in the usual manner, as follows :

1	BC	1	bc
AB	ABC	ab	abc
AC	ACB	ac	acb

Since the substitutions of a row transform the corresponding functions in the same way and the rows of the two rectangles evidently correspond in order, we may effect the required transformation by any two substitutions such that one belongs to the first and the other to the second of the following two rows :

$$AB.ab,\ AB.abc,\ ABC.ab,\ ABC.abc$$

$$AC.ac,\ AC.acb,\ ACB.ac,\ ACB.acb$$

If we use the last one of each of these rows we have the rule frequently given in the text-books, viz., "The corresponding formulas for the other two angles may be obtained from this by the cyclical interchange of the letters."

The given formula might also be studied by employing a single group in place of two. The most convenient group is the intransitive group of degree six and order 36 which is obtained by multiplying the symmetric group of the angles into the symmetric group of the sides.* Since the given formula is transformed into itself only by the following four substitutions of this group

$$1,\ BC,\ bc,\ BC.bc,$$

it is a nine-valued function with respect to this group.† Since substitutions of this group transform the first member of the given formula into its three values without affecting the second member, these nine values may be arranged into three triplets, each of which has the same second member. By very simple trials we can show that six of these relations are absurd. Since three must be true the remaining relations are the required formulas.

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*Cf. This journal, Vol. II, page 307.

†This may be proved in exactly the same way as Lagrange's theorem was proved.

Since similar remarks apply to a large number of the other Trigonometry formulas, it is clear that these formulas can be discussed in a more general and more definite manner by presupposing a thorough knowledge of the given elementary principles of substitution groups.

It is also easy to show that many problems of factoring can be discussed more completely by presupposing a knowledge of these groups. The following is a very simple illustration :

$$a^2-b^2-c^2+2bc=(a+b-c)(a-b+c) \ldots\ldots\ldots\ldots\ldots C.$$

The expression belongs to the group (bc) and is therefore a three-valued function; its factors belong to the groups (ab) and (ac) respectively and are therefore also three-valued functions. Hence C indicates an equality between a given three-valued function and the product of two other three-valued functions. These functions belong to three distinct groups. Arranging the substitutions of these groups in the usual manner, we have

1	bc	1	ab	1	ac
ab	abc	ac	abc	ab	acb
ac	acb	bc	acb	bc	abc

The three values of the given expression* may be obtained by applying to it one substitution from each of the three rows of the first rectangle, e. g., the first column. The factors of these transforms may evidently be found by applying the same substitutions to the given factors. Since ab and ac transform one of the factors into itself it follows that the three conjugate expressions contain only three distinct linear factors, viz., the three values of any one of them.

These observations indicate how we may readily determine the total number of substitutions by means of which the factors of all the conjugates of a given expression may be found from those of the given expression. They have brought us in contact with, at least, three important questions, viz. :

1. What relations exist between the factors of a system of conjugate expression ?

2. What relations exist between the groups of the factors and the group of the expression ?

3. To what extent may these relations be utilized in the process of factoring ?

*The same idea is expressed by "the three conjugates of the given expression" or by "the three transforms of the given expression."

THE BINOMIAL THEOREM.

By G. B. M. ZERR, A. M., Ph. D., Texarkana, Texas.

I use the following rule for expanding all binomials, whether the exponent is integral or fractional, positive or negative.

The number of terms of a binomial expansion is one more than the exponent when the exponent is a positive integer, otherwise the number of terms is infinite. For the first term of the expansion, raise the first term of the binomial to the required power. For any other term of the expansion, multiply the preceding term by the second term of the binomial, and this product by the exponent of the power diminished by two less than the number of terms from the beginning, divide this product by the product of the first term of the binomial into one less than the number of terms from the beginning, always observing the proper algebraic signs of the binomial terms.

$$(ax+by)^m=(ax)^m+\frac{m(ax)^m by}{ax}+\frac{m(m-1)(ax)^m(by)^2}{1.2.(ax)^2}+\ldots\ldots$$

$$+\ldots\ldots+\frac{m(m-1)(m-2)\ldots\ldots(m-r+2)(ax)^m(by)^{r-1}}{1.2.3\ldots\ldots(r-1)(ax)^{r-1}}+\ldots\ldots (A).$$

(A) gives the expansion without reducing the terms.

(1). To expand $(3x\pm4y)^5$.

1st term $=(3x)^5=243x^5$; 2nd term $=\dfrac{243x^5\times(\pm4y)\times5}{3x}=\pm1620x^4y$;

$$\text{3rd term}=\frac{\pm1620x^4y\times(\pm4y)\times4}{2.3x}=4320x^3y^2;$$

$$\text{4th term}=\frac{4320x^3y^2\times(\pm4y)\times3}{3.3x}=\pm5760x^2y^3;$$

$$\text{5th term}=\frac{\pm5760x^2y^3\times(\pm4y)\times2}{4.3x}=3840xy^4;$$

$$\text{6th term}=\frac{3840xy^4\times(\pm4y)\times1}{5.3x}=\pm1024y^5.$$

$$\therefore\ (3x\pm4y)^5=243x^5\pm1620x^4y+4320x^3y^2\pm5760x^2y^3+3840xy^4\pm1024y^5.$$

(2). To expand $(a^2+2b)^7$.

1st term $=(a^2)^7=a^{14}$; 2nd term $=\dfrac{a^{14}.2b.7}{a^2}=14a^{12}b$;

3rd term $=\dfrac{14a^{12}b.2b.6}{2.a^2}=84a^{10}b^2$; 4th term $=\dfrac{84a^{10}b^2.2b.5}{3.a^2}=280a^8b^3$;

5th term $=\dfrac{280a^8b^3.2b.4}{4.a^2}=560a^6b^4$; 6th term $=\dfrac{560a^6b^4.2b.3}{5.a^2}=672a^4b^5$;

7th term $=\dfrac{672a^4b^5.2b.2}{6.a^2}=448a^2b^6$; 8th term $=\dfrac{448a^2b^6.2b.1}{7.a^2}=128b^7$.

$$\therefore (a^2+2b)^7=a^{14}+14a^{12}b+84a^{10}b^2+280a^8b^3 \\ +560a^6b^4+672a^4b^5+448a^2b^6+128b^7.$$

(3). To expand $(2+x)^{-3}$.

1st term $=(2)^{-3}=\frac{1}{8}$; 2nd term $=\frac{1}{8}\times\dfrac{x\times(-3)}{2}=-\dfrac{3x}{16}$;

3rd term $=-\dfrac{3x}{16}\times\dfrac{x\times(-4)}{2.2}=\dfrac{3x^2}{16}$; 4th term $=\dfrac{3x^2}{16}\times\dfrac{x\times(-5)}{3.2}=-\dfrac{5x^3}{32}$;

5th term $=-\dfrac{5x^3}{32}\times\dfrac{x\times(-6)}{4.2}=\dfrac{15x^4}{128}$.

$$\therefore (2+x)^{-3}=\frac{1}{8}-\frac{3x}{16}+\frac{3x^2}{16}-\frac{5x^3}{32}+\frac{15x^4}{128}-\ldots\ldots$$

(4). To expand $(1+\frac{2x}{3})^{\frac{3}{2}}$.

1st term $=(1)^{\frac{3}{2}}=1$; 2nd term $=\dfrac{1.\frac{2x}{3}.\frac{3}{2}}{1}=x$;

3rd term $=\dfrac{x.\frac{2x}{3}.\frac{1}{2}}{2.1}=\frac{1}{6}x^2$; 4th term $=\dfrac{\frac{1}{6}x^2.\frac{2x}{3}.(-\frac{1}{2})}{3.1}=-\frac{1}{54}x^3$;

5th term $=\dfrac{-\frac{1}{54}x^3.\frac{2x}{3}.(-\frac{3}{2})}{4.1}=\frac{1}{216}x^4$.

$\therefore\ (1+\frac{2x}{3})^{\frac{3}{2}}=1+x+\frac{1}{6}x^2-\frac{1}{54}x^3+\frac{1}{216}x^4-\ \ldots\ldots$

(5). To expand $(8+12a)^{\frac{2}{3}}$.

$$\text{1st term}=(8)^{\frac{2}{3}}=4;\ \text{2nd term}=\frac{4.12a.\frac{2}{3}}{8}=4a\ ;$$

$$\text{3rd term}=\frac{4a.12a.(-\frac{1}{3})}{2.8}=-a^2;\ \text{4th term}=\frac{-a^2.12a.(-\frac{4}{3})}{3.8}=\frac{2a^3}{3};$$

$$\text{5th term}=\frac{\frac{2a^3}{3}.12a.(-\frac{7}{3})}{4.8}=-\frac{7a^4}{12}.$$

$\therefore\ (8+12a)^{\frac{2}{3}}=4+4a-a^2+\frac{2}{3}a^3-\frac{7}{12}a^4+\ldots\ldots$

(6). To expand $(4a-8x)^{-\frac{1}{2}}$.

$$\text{1st term}=(4a)^{-\frac{1}{2}}=\frac{1}{2a^{\frac{1}{2}}};\ \text{2nd term}=\frac{1}{2a^{\frac{1}{2}}}.\ \frac{(-8x)(-\frac{1}{2})}{4a}=\frac{x}{2a^{\frac{3}{2}}};$$

$$\text{3rd term}=\frac{x}{2a^{\frac{3}{2}}}.\ \frac{(-8x)(-\frac{3}{2})}{2.4a}=\frac{3x^2}{4a^{\frac{5}{2}}};\ \text{4th term}=\frac{3x^2}{4a^{\frac{5}{2}}}.\ \frac{(-8x)(-\frac{5}{2})}{3.4a}=\frac{5x^3}{4a^{\frac{7}{2}}};$$

$$\text{5th term}=\frac{5x^3}{4a^{\frac{7}{2}}}.\ \frac{(-8x)(-\frac{7}{2})}{4.4a}=\frac{35x^4}{16a^{\frac{9}{2}}}.$$

$$\therefore\ (4a-8x)^{-\frac{1}{2}}=\frac{1}{2a^{\frac{1}{2}}}+\frac{x}{2a^{\frac{3}{2}}}+\frac{3x^2}{4a^{\frac{5}{2}}}+\frac{5x^3}{4a^{\frac{7}{2}}}+\frac{35x^4}{16a^{\frac{9}{2}}}+\ldots\ldots$$

$$=\frac{1}{2a^{\frac{1}{2}}}\left(1+\frac{x}{a}+\frac{3x^2}{2a^2}+\frac{5x^3}{4a^3}+\frac{35x^4}{8a^4}+\ldots\ldots\right).$$

The r^{th} term in (A) is $\dfrac{m(m-1)(m-2)\ldots\ldots(m-r+2)(ax)^m(by)^{r-1}}{1.2.3\ldots\ldots(r-1)(ax)^{r-1}}$.

(7). Find the 4th term of $(\frac{a}{3}+9b)^{10}$.

$$m=10,\ r=4.\quad \therefore\ \text{4th term}=\frac{10\times9\times8\times(\frac{a}{3})^{10}(9b)^3}{1.2.3.(\frac{a}{3})^3}=40a^7b^3.$$

(8). Find the 28th term of $(5x+8y)^{30}$.
$m=30$, $r=28$.

$$\therefore \text{28th term}=\frac{30.29.28\ldots\ldots6.5.4.(5x^{30}(8y)^{27}}{1.2.3\ldots\ldots26.27.(5x)^{27}}=\frac{\underline{|30}}{\underline{|27}\;\;\underline{|3}}(5x)^3(8y)^{27}.$$

(9). Find the 8th term of $(1+2x)^{-\frac{1}{2}}$.
$m=-\frac{1}{2}$, $r=8$.

$$\therefore \text{8th term}=\frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})(-\frac{7}{2})(-\frac{9}{2})(-\frac{11}{2})(-\frac{13}{2})(1)^{-\frac{1}{2}}(2x)^7}{1.2.3.4.5.6.7.(1)^7}=-\frac{429x^7}{16}.$$

(10). Find the 10th term of $(1+3a^2)^{\frac{16}{3}}$.
$m=\frac{16}{3}$, $r=10$.

$$\therefore \text{10th term}=\frac{\frac{16}{3}.\frac{13}{3}.\frac{10}{3}.\frac{7}{3}.\frac{4}{3}.\frac{1}{3}(-\frac{2}{3}).(-\frac{5}{3}).(-\frac{8}{3})(1)^{\frac{16}{3}}(3a^2)^9}{1.2.3.4.5.6.7.8.9.(1)^9}=-\frac{1040a^{18}}{81}.$$

(11). Find the 5th term of $(3a-2b)^{-1}$.
$m=-1$, $r=5$.

$$\therefore \text{5th term}=\frac{(-1)(-2)(-3)(-4)(3a)^{-1}(2b)^4}{1.2.3.4.(3a)^4}=\frac{16b^4}{243a^5}.$$

These are enough examples to illustrate both the rule and the general term.

I have used this method with my classes for several years and find it easier and better than any other method I have ever used. I have never seen this method in this form. If any of the readers of the MONTHLY have ever seen it, I would be pleased to know where to find it.

A GEOMETRICAL PROOF THAT $0\times\infty$ IS INDETERMINATE.

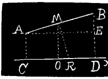
By B. F. FINKEL, A. M., Professor of Mathematics and Physics in Drury College, Springfield, Missouri.

The Proof that I shall offer is not new perhaps, but I have never seen it in print, and for that reason I shall give it in the MONTHLY.

The proof follows as a corollary of the following

PROPOSITION: *The area of the surface generated by a straight line revolving about an axis in its plane is equal to the product of the projection of the line on the axis by the circumference whose radius is a perpendicular erected at the middle point of the line and terminated by the axis.*

Let AB be the straight line revolved about CD as an axis. When AB is not parallel to the axis CX, the surface generated by AB is the convex surface of the frustum of a cone.



$\therefore$ Area generated by $AB = AB\times 2\pi MO$.

But $MO : AE :: MR : AB$, or

$$AB\times MO = AE\times MR = CD\times MR.$$

$\therefore$ Area generated by $AB = CD\times 2\pi MR$. Now if AB is made to approach perpendicularity, MR will approach parallelism to CX, and, in consequence, CD will approach 0 as its limit and MR will approach ∞ as its limit. Hence, in the limit, we have

$$\text{area generated by } AB = CD\times 2\pi MR = 0\times 2\pi\times\infty.$$

But area generated by AB when AB is perpendicular to CX is $\pi(BC^2-AC^2)$. Hence, $\pi(BC^2-AC^2)=0\times 2\pi\times\infty$, or $BC^2-AC^2=2\times 0\times\infty=0\times\infty$. When $AC=0$, we have $0\times\infty=BC^2$. Now BC is entirely arbitrary. Hence, $0\times\infty$ is indeterminate. But when BC is a definite quantity, as for example 3, then $0\times\infty$ has the definite value 9.

The fundamental type of symbols of indetermination is $\frac{0}{0}$, and to this type $0\times\infty$ may be reduced. Thus, $0\times\infty=0\times\frac{1}{\frac{1}{\infty}}=\frac{0\times 1}{0}=\frac{0}{0}$. The indeterminate form, $\frac{\infty}{\infty}$, $=\frac{\frac{1}{\infty}}{\frac{1}{\infty}}=\frac{0}{0}$. Also $\infty-\infty=\frac{1}{\frac{1}{\infty}}-\frac{1}{\frac{1}{\infty}}=\frac{1}{0}-\frac{1}{0}=\frac{0}{0}$; $0^0=0^a\div 0^a=\frac{0^a}{0^a}=\frac{0}{0}$; $\infty^0=\infty^a\div\infty^a=\frac{\infty}{\infty}=\frac{\frac{1}{\infty}}{\frac{1}{\infty}}=\frac{0}{0}$.

When these forms occur as the answers of problems, they have, in general, perfectly definite values, and these definite values must be found. But when these forms stand apart from the consideration of problems, they are perfectly meaningless,

Drury College, September 14, 1896.

ARITHMETIC.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

60. Proposed by J. K. ELLWOOD, A. M., Principal of Colfax School, Pittsburg, Pennsylvania.

A pipe 1 foot long and 27-32 inch in diameter has a half-inch orifice and weighs $1\frac{3}{4}$ pounds. What is the diameter of a pipe the same length and orifice, but weighing 41 ounces?

I. Solution by F. M. McGAW, A. M., Professor of Mathematics, Bordentown Military Institute, Bordentown, New Jersey.

Let V_1 = volume of "solid" pipe.

Let V_2 = volume of bore.

Then $V_1 - V_2$ = volume of metal $=\pi(\frac{1419}{1024})$ cubic inches.

Since weights are proportional to volumes, $\pi(\frac{1419}{1024}) : V_3 = 28 : 41$, where V_3 = volume of required size of pipe.

Add to this volume of bore $= V_2$, and we have,

$$V_3 + V_2 = V_4 = \text{new "solid" pipe} = \pi(\tfrac{79683}{28672}) \text{ cubic inches.}$$

$$\text{Hence } R = \{\pi(\tfrac{79683}{28672}) \div \pi.12\}^{\frac{1}{2}} = \tfrac{431}{896}.192 \text{ and } D = .9625 \text{ inches.}$$

II. Solution by EDWARD R. ROBBINS, Master in Mathematics and Physics, Lawrenceville School, Lawrenceville, New Jersey.

The volumes of the two pipes will have the same ratio as their weights.

$$\text{Hence, } \frac{\pi l\left[\left(\frac{D}{2}\right)^2 - \left(\frac{d}{2}\right)^2\right]}{\pi l\left[\left(\frac{D'}{2}\right)^2 - \left(\frac{d}{2}\right)^2\right]} = \frac{w}{w'}; \text{ or } \frac{D^2 - d^2}{D'^2 - d^2} = \frac{w}{w'},$$

where the D's represent the diameters of the pipes, and d the common diameter of their *orifices*. From this

$$D'=\sqrt{\frac{w'D^2-w'd^2+wd^2}{w}}=\sqrt{\frac{w'}{w}(D^2-d^2)+d^2}.$$

$$=\sqrt{\tfrac{41}{28}[(\tfrac{27}{32})^2-(\tfrac{1}{2})^2]+\tfrac{1}{4}}=\sqrt{\tfrac{26561}{28672}}=.96248+ \text{ inches}.$$

Also solved by *G. B. M. ZERR*, *H. C. WILKES*, and *J. SCHEFFER*.

61. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

Insured my store for a/bth$=\frac{3}{4}$th of its value, at $r=1\frac{1}{4}\%$; but soon afterward the store was burned down, and my loss over the insurance was $\$L=\$4150$. What was the value of my store?

I. Solution by HON. JOSIAH H. DRUMMOND, LL. D., Portland, Maine.

Construing the terms of this question as they are used in legal and insurance circles the solution is $\$4,150\times4=\$16,600$.

But the proposer evidently intends to reckon the premium paid as a part of the "loss."

Then for every \$4.00 of value \$3.00 was insured at a cost of 3.75 cents, leaving \$1.0375 of loss.

Hence 1.0375 : 4 :: 4150 : 16,000.

II. Solution by J. SCHEFFER, A. M., Hagerstown, Maryland.

The value of the policy is $\frac{a}{b}\cdot\frac{r}{100}x$, x representing the value of the store.

We have, therefore, obviously,

$$x\left[1-\frac{a}{b}\left(1-\frac{r}{100}\right)\right]=L,\ \therefore x=L\div\left[1-\frac{a}{b}\left(1-\frac{r}{100}\right)\right].$$

Substituting numerical value, we find $x=\$16,000$.

III. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas; P. S. BERG, Larimore, North Dakota; and A. P. READ, A. M., Clarence, Missouri.

Let $s=$ value of store.

$$\text{Then } s-\frac{as}{b}=\left(\frac{b-a}{b}\right)s;\ \frac{r}{100}\times\frac{as}{b}=\frac{ars}{100b}.\ \therefore\left(\frac{b-a}{b}+\frac{ar}{100b}\right)s=L.$$

$$\therefore s=\frac{100bL}{100b-100a+ar}=\frac{400(4150)}{100+\frac{15}{4}}=\$16,000.$$

Also solved by *EDWARD R. ROBBINS*, and *F. M. McGAW*.

PROBLEMS.

65. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

Bought April 4, 1894, 250 yards of broadcloth at \$5.37½ per yard, less 12½ and 10% discount for cash payment. Sold September 5, 1894, at 15, 10, and 5% on *quoted price*, the cloth; and in settlement received a 90-day note which I had discounted at 5½%, October 19, 1894, by the First National Bank of Baltimore, Maryland. Reckoning 6% interest on the *money invested* in the cloth, what is the profit made?

66. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

Brown adds m=10% of water to the pure wine he buys, and then sells the mixture at a price n=10% greater than the cost price of the pure wine. What is his rate per cent. of profit?

ALGEBRA.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

62. Proposed by Professor C. E. WHITE, A. M., Trafalgar, Indiana.

Prove that every algebraic equation can be transformed into another equation of the same degree, but which wants its n^{th} term.

I. Solution by HENRY HEATON, M. Sc., County Surveyor, Atlantic, Iowa.

To illustrate, let $x^4+ax^3+bx^2+cx+d=0$ be any equation of the fourth degree. Put $x=y+p$; then the equation becomes

$$y^4+(4p+a)y^3+(6p^2+3ap+b)y^2+(4p^3+3ap^2+2bp+c)y+p^4+ap^3+bp^2+cp+d=0.$$

Since we are at liberty to give p any value, we may give it the value that will make $4p+a=0$ or $-a/4$; then will the coefficient of y^3 disappear. It is also evident that we may give p such a value that any desired coefficient will disappear. It is also evident that to find the desired value of p by this method requires for the second term, the solution of an equation of the first degree; for the third term, the solution of an equation of the second degree, etc. It is further evident that this is true without regard to the degree of the original equation.

II. Solution by BENJ. F. YANNEY, A. M., Professor of Mathematics in Mount Union College, Alliance, Ohio.

If not already so, any equation of the n^{th} degree may be reduced to the form $x^n+Ax^{n-1}+Bx^{n-2}+\ldots\ldots+L=0$. Now, by putting for x, $x+a$, we obtain a new equation whose roots differ from the corresponding roots of the given equation by a, (and whose degree, therefore, is still the n^{th}) viz.:

$$x^n+(na+A)x^{n-1}+\left(\frac{n(n-1)}{|2}a^2+(n-1)Aa+B\right)x^{n-2}$$

$$+\ldots\ldots+(a^n+Aa^{n-1}+Ba^{n-2}+\ldots\ldots+L=0.$$

As a is an arbitrary constant, it may be selected so that $(na+A)=0$, or

$$\left(\frac{n(n-1)}{|2}a^2+(n-1)Aa+B\right)=0,$$

or any coefficient, except the first, $=0$. Hence, any term, except the first, may thus be removed.

III. Solution by O. W. ANTHONY, M. Sc., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

Every algebraic equation may be written

$$X^k-\Sigma\alpha.X^{k-1}+\Sigma\alpha\beta.X^{k-2}\ldots\ldots=0.$$

The coefficient of the n^{th} term will be $\Sigma\alpha\beta\gamma\ldots\ldots$to $n-1$ factors. Now in place of X write $X+h$; then α, β, γ, etc., will be changed into $\alpha+h$, $\beta+h$, $\gamma+h$, etc. The coefficient of n^{th} term will then be $\pm\Sigma(\alpha+h)(\beta+h)(\gamma+h)\ldots\ldots$to $n-1$ terms. If we equate this to zero, we may consider it an equation of degree $n-1$ in h. This will give $n-1$ values of h. Therefore there are $n-1$ transformations which will make the n^{th} term vanish. Consider the first term, $n-1$; there are in that case no transformations.

Also solved by *PROF. E. W. MORRELL.*

63. Proposed by J. A. CALDERHEAD, A. B., Professor of Mathematics in Curry University, Pittsburg, Pennsylvania.

Given $x^2+x\sqrt{xy}=10$, and $y^2+y\sqrt{xy}=20$ to find x and y by quadratics.

I. Solution by E. L. BROWN, A. M., Professor of Mathematics, Capital University, Columbus, Ohio; HENRY HEATON, M. Sc., Atlantic, Iowa; and G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Factoring, we have $x^{\frac{3}{2}}(x^{\frac{1}{2}}+y^{\frac{1}{2}})=10$, $y^{\frac{3}{2}}(x^{\frac{1}{2}}+y^{\frac{1}{2}})=20$.

$$\therefore y^{\frac{3}{2}}/x^{\frac{3}{2}}=2,\ y^{\frac{3}{2}}=2x^{\frac{3}{2}}.\quad \therefore y=\sqrt[3]{4}x.$$

$\therefore y^{\frac{1}{2}}=\pm x^{\frac{1}{2}}\sqrt[3]{2}$, this in either equation gives

$$x^2(1\pm\sqrt[3]{2})=10,\quad \therefore x=\pm\sqrt{\frac{10}{1\pm\sqrt[3]{2}}},\ y=\pm\sqrt[3]{4}\sqrt{\frac{10}{1\pm\sqrt[3]{2}}}.$$

II. Solution by J. K. ELLWOOD, A. M., Principal of Colfax School, Pittsburg, Pennsylvania; COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tennessee; and M. A. GRUBER, A. M., War Department, Washington, D. C.

Factoring the given equations, we obtain

$$x\sqrt{x}(\sqrt{x}+\sqrt{y})=10=a,\ldots\ldots\ldots\ldots(1),\quad y\sqrt{y}(\sqrt{x}+\sqrt{y})=20=b,\ \ldots\ldots\ldots\ldots(2).$$

$(1)\div(2)$ gives $\dfrac{x\sqrt{x}}{y\sqrt{y}}=\dfrac{a}{b}$. Squaring and reducing, we get

$$y=\frac{x\sqrt[3]{b^2}}{\sqrt[3]{a^2}},\ \text{and}\ \sqrt{xy}=\frac{x\sqrt[3]{b}}{\sqrt[3]{a}}.$$

Substituting in first given equation, we have $x^2+\dfrac{x^2\sqrt[3]{b}}{\sqrt[3]{a}}=a$;

$$\text{whence } x=\pm\left(\frac{a\sqrt[3]{a}}{\sqrt[3]{a}+\sqrt[3]{b}}\right)^{\frac{1}{2}}=\pm\left(\frac{10}{1+\sqrt[3]{2}}\right)^{\frac{1}{2}},$$

$$\text{and } y=\pm\left(\frac{b\sqrt[3]{b}}{\sqrt[3]{a}+\sqrt[3]{b}}\right)^{\frac{1}{2}}=\pm\left(\frac{20\sqrt[3]{2}}{1+\sqrt[3]{2}}\right)^{\frac{1}{2}}.$$

III. Solution by BENJ. F. YANNEY, A. M., Professor of Mathematics, Mount Union College, Alliance, Ohio; and Prof. E. W. MORRELL, Montpelier Seminary, Montpelier, Vermont.

The given equations may be written,

$$x\sqrt{xy}=10-x^2\ldots\ldots\ \ldots\ldots(1),\quad y\sqrt{xy}=20-y^2\ldots\ldots\ldots\ldots(2).$$

$(1)\times(2)$, $x^2y^2=200-20x^2-10y^2+x^2y^2$. $\quad\therefore 2x^2+y^2=20\ldots\ldots\ldots\ldots\ldots(3)$.

From (2) and (3), $2x^2=y\sqrt{xy}$. $\therefore y=x\sqrt[3]{4}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$.

$$(4)\text{ in }(1),\ x=\pm\sqrt{\frac{10}{1+\sqrt[3]{2}}}.\quad \therefore y=\pm\sqrt{\frac{20\sqrt[3]{2}}{1+\sqrt[3]{2}}}.$$

IV. Solution by J. H. DRUMMOND, LL. D., Portland, Maine; A. H. HOLMES, Brunswick, Maine; and O. W. ANTHONY, M. Sc., New Windsor College, New Windsor, Maryland.

Let $y=v^2x$, then $x^2(1+v)=a=10$, and $v^3x^2(1+v)=b=20$.

$$\therefore v=\sqrt[3]{\frac{b}{a}},\ \text{and}\ x=\pm\frac{a^{\frac{2}{3}}}{\sqrt{a^{\frac{1}{3}}+b^{\frac{1}{3}}}},\ =\pm\left(\frac{10}{1+\sqrt[3]{2}}\right)^{\frac{1}{2}}.$$

$$y=\pm\frac{b^{\frac{2}{3}}}{a^{\frac{1}{3}}+b^{\frac{1}{3}}},\ =\pm\left(\frac{20\sqrt[3]{2}}{1+\sqrt[3]{2}}\right)^{\frac{1}{2}}.$$

V. [illegible] A. HOBBS, A. M., Master of Mathematics in the Belmont School, Belmont, Massachusetts.

$$x^2+x^{\frac{3}{2}}y^{\frac{1}{2}}=10,\ y^2+x^{\frac{1}{2}}y^{\frac{3}{2}}=20.\quad \text{Let } y=vx.$$

$$\text{Then } x^2+v^{\frac{1}{2}}x^2=10,\ v^2x^2+v^{\frac{3}{2}}x^2=20.$$

$$\therefore\ x^2=\frac{10}{1+v^{\frac{1}{2}}},\ \text{and } x^2=\frac{20}{v^2+v^{\frac{3}{2}}}.\quad \therefore\ \frac{10}{1+v^{\frac{1}{2}}}=\frac{20}{v^2+v^{\frac{3}{2}}}.$$

Dividing by 10, and clearing of fractions, $v^{\frac{3}{2}}=2$, $v=2^{\frac{2}{3}}$.

$$\therefore\ x^2=\frac{10}{1+2^{\frac{1}{3}}},\ x=\sqrt{\frac{10}{1+\sqrt[3]{2}}}.\quad y=2^{\frac{2}{3}}\sqrt{\frac{10}{1+\sqrt[3]{2}}}=\sqrt{\frac{20\sqrt[3]{2}}{1+\sqrt[3]{2}}}.$$

VI. Solution by J. W. WATSON, Middle Creek, Ohio; and H. C. WILKES, Skull Run, West Virginia.

Put $x=m^2$, $y=n^2$. Then, the given equations become, after factoring,

$m^3(m+n)=10$ ....(1), and $n^3(m+n)=20$........(2). Whence $n=m\sqrt[3]{2}$.

Then in (1) $m^3(m+m\sqrt[3]{2})=10$, or $m^4(1+\sqrt[3]{2})=10$.

$$\therefore\ m^4=\frac{10}{1+\sqrt[3]{2}},\ \text{and } m^2=\pm\sqrt{\frac{10}{1+\sqrt[3]{2}}},\ =x.$$

Also, $n^2,\ =y,\ =\pm\sqrt{\dfrac{20\sqrt[3]{2}}{1+\sqrt[3]{2}}}$.

GEOMETRY

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

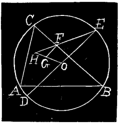
SOLUTIONS OF PROBLEMS.

56. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

The locus of the centers of the isogonal transformations of all the diameters of the circumcircle of any triangle is the nine-points circle. *Brocard.*

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science in Texarkana College, Texarkana, Arkansas-Texas.

Let O and H be the circum and ortho-centers respectively, of the triangle ABC. Draw the diameter DE, connect E and H, and from F the mid-point of EH draw FG parallel to OE.



Now H and O are inverse points.

G is the mid-point of HO and $GF=\frac{1}{2}OE=$a constant.

$\therefore$ G is the center and GF the radius of the nine-point circle.

$\therefore$ The locus of F is the nine-point circle.

II. Solution by the PROPOSER.

Let $l\alpha+m\beta+n\gamma=0$..........(1) be any diameter. The isogonal transformation of (1) is

$$\frac{l}{\alpha}+\frac{m}{\beta}+\frac{n}{\gamma}=0 \qquad \text{(2)}.$$

Now (1), passing through the center of the circumcircle, the coordinates of which are proportional to $\cos A$, $\cos B$, $\cos C$, gives the relation

$$l\cos A+m\cos B+n\cos C=0 \qquad \text{(3)}.$$

Also, the center of (2), which is an equilateral hyperbola, with condition (3), is given by

$$\frac{l}{n}=\frac{-a\alpha^2+b\alpha\beta+c\alpha\gamma}{b\beta\gamma-c\gamma^2+a\alpha\gamma},\quad \frac{m}{n}=\frac{a\alpha\beta-b\beta^2+c\beta\gamma}{b\beta\gamma-c\gamma^2+a\alpha\gamma} \qquad \text{(4)}.$$

Dividing (3) by n, and substituting equations (4), and reducing,

$$a\beta\gamma+b\alpha\gamma+c\alpha\beta-a\alpha^2\cos A-b\beta^2\cos B-c\gamma^2\cos C=0 \qquad \text{(5)},$$

the nine-points circle.

57. Proposed by J. OWEN MAHONEY, B. E., Graduate Fellow and Assistant in Mathematics, Vanderbilt University, Nashville, Tennessee.

Show that pairs of points, on a straight line, may be so related harmonically that a pair of real points will be harmonic with regard to a pair of imaginary points, and by this means prove that there are an indefinite number of conjugate pairs of imaginary points on a real line.

I. Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy in Ohio University, Athens, Ohio.

If the four points be A, B, C, D, and the axis of x coincide with the given straight line, A, B may be supposed given by

$$\alpha x^2+2\beta x+\gamma=0 \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (1),$$

$$\text{or } x=\frac{-\beta\pm\sqrt{\beta^2-\gamma^2}}{\alpha} \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (2),$$

$$\text{and } C,\ D, \text{ by } \alpha' x^2+2\beta' x+\gamma'=0 \ldots \ldots \ldots \ldots \ldots \ldots (3).$$

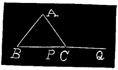
Now as long as γ exceeds β, (2) gives imaginary values for x, and so for a like pair of values for (3), which does not violate the condition

$$\alpha\gamma'+\alpha'\gamma=2\beta\beta' \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (4),$$

any number of values of β, γ in (2) always being consistent with (4).

II. Solution by JOHN B. FAUGHT, A. M., Instructor in Mathematics, Indiana University, Bloomington, Indiana.

Using trilinear coordinates, take B and C for the two real points on the real line $\alpha=0$, i. e., $b\beta+c\gamma=2\triangle$. $B^2+K^2\gamma^2=0$, is the equation of two lines through A; that is $\beta+Ki\gamma=0$, and $\beta-Ki\gamma=0$. These lines form with $\beta=0$ (AC) and $\gamma=0$ (AB) a harmonic pencil, and hence intersect BC in two points forming with B and C a harmonic range.



Moreover these lines are imaginary for all real values of K and hence must intersect BC in imaginary points, otherwise they would contain two real points, which is impossible.

The coordinates of the points of intersections of these imaginary lines may be found by solving with $b\beta+c\gamma=2\triangle$. Thus $\beta=-Ki\gamma$ gives

$$(c-bKi)\gamma=2\triangle \text{ and } \gamma=\frac{2\triangle c}{c^2+b^2K^2}+\frac{2\triangle Kb}{c^2+b^2K^2}i$$

$$\text{and } \beta=\frac{2\triangle K^2b}{c^2+b^2K^2}-\frac{2\triangle Kc}{c^2+b^2K^2}i, \text{ and } \beta=Ki\gamma, \text{ gives}$$

$$\gamma=\frac{2\triangle c}{c^2+b^2K^2}-\frac{2\triangle Kb}{c^2+b^2K^2}i, \text{ and } \beta=\frac{2\triangle K^2b}{c^2+b^2K^2}+\frac{2\triangle Kc}{c^2+b^2K^2}i.$$

If P and Q denote the imaginary points of intersection, we see that their coordinates are conjugates. These points are called "conjugués harmoniques" with respect to B and C, by M. Chasles.

It is evident that by giving different values to K an infinite number of such points can be found.

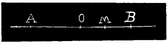
III. Solution by the PROPOSER.

The roots of $ax^2+2bx+c=0$ and $a'x^2+2b'x+c'=0$ will be harmonic if $ac'+a'c-2bb'=0$ (see Scott's Geometry, page 45).

Let $x^2=p^2$ give the points A and B. Let $x=OM=K<(OB=p)$ be midway between the other points, P and Q. The equation giving P and Q is

$$a'x^2+2b'x+c'=0, \text{ with the conditions } \frac{b'}{a}=-K, \text{ and } c'-p^2a'=0,$$

$$\text{or } x^2-2Kx+p^2=0.$$



But since $K<p$, $K^2-p^2<0$, the roots of this equation are imaginary, and since there are an indefinite number of values for $K<p$, there will be an indefinite number of pairs of imaginary points on the line harmonic with the given real pair. (Scott's Geometry, page 45.)

Solved in a similar manner by *G. B. M. ZERR*.

PROBLEMS.

63. Proposed by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, P. O. University, Mississippi.

Prove, analytically :—A rectangular hyperbola cannot be cut from a right circular cone unless the angle at its vertex is greater than a right angle.

64. Proposed by WILLIAM E. HEAL, Member of the London Mathematical Society and Treasurer of Grant County, Marion, Indiana.

Let the bisectors of the angles A, B, C of a triangle meet the sides opposite A, B, C in A', B', C'. Let AA', BB', CC' meet the sides of the triangle $A'B'C'$ in A'', B'', C''. Let this process continue indefinitely. Express the sides and angles of the triangle $A^{(m)}B^{(m)}C^{(m)}$ in terms of the sides and angles of the original triangle ABC.

MECHANICS.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

34. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics and Astronomy, New Windsor College, New Windsor, Maryland.

A particle is placed within a thin cylindrical shell without ends. Find the resultant attraction, the cylinder being composed of matter attracting according to the laws of nature.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science in Texarkana College, Texarkana, Arkansas-Texas.

Let $r=2a\sin\theta$ be the equation to the cylinder, so that the origin is in the surface of the cylinder at one end, then $y=r\sin\theta=2a\sin^2\theta$, $z=r\cos\theta=2a\sin\theta\cos\theta$, l=length of cylinder, (x, y, z) coordinates of any point in the shell, ρ=density, k=thickness of shell. It is always possible to take the axes of coordinates so that the particle will lie in the plane of the axis of y; let (m, n, o) be the coordinates of the particle, mass unity.

$$ds=2adxd\theta,\ p=\sqrt{(x-m)^2+(y-n)^2+z^2}=\sqrt{(x-m)^2+n^2-4a(n-a)\sin^2\theta},\ n>a.$$

This will give attraction for all possible positions of the particle. For $n<a$,

$$p=\sqrt{(x-m)^2+(2a-n)^2-4a(a-n)\sin^2(\tfrac{1}{2}\pi-\theta)},$$

and the solution would be the same as for $n>a$.

$$\text{Let } \frac{4a(n-a)}{m^2+n^2}=b^2,\quad \frac{4a(n-a)}{(l-m)^2+n^2}=c^2,\quad \frac{4a(n-a)}{n^2}=d.$$

Resolving the attractions parallel to the axes, we easily get

$$X=2a\rho k\int_0^\pi\int_0^l\frac{(x-m)d\theta dx}{\{(x-m)^2+n^2-4a(n-a)\sin^2\theta\}^{\frac{3}{2}}}$$

$$=2a\rho k\int_0^\pi\left\{\frac{1}{\sqrt{m^2+n^2-4a(n-a)\sin^2\theta}}-\frac{1}{\sqrt{(l-m)^2+n^2-4a(n-a)\sin^2\theta}}\right\}d\theta$$

$$=\frac{2a\rho k}{\sqrt{m^2+n^2}}E_0^\pi(b,\theta)-\frac{2a\rho k}{\sqrt{(l-m)^2+n^2}}E_0^\pi(c,\theta).$$

$$Y=2a\rho k\int_0^\pi\int_0^l\frac{(2a\sin^2\theta-n)d\theta dx}{\{(x-m)^2+n^2-4a(n-a)\sin^2\theta\}^{\frac{3}{2}}},$$

$$=2a\rho k\int_0^\pi\left\{\frac{l-m}{\sqrt{(l-m)^2+n^2-4a(n-a)\sin^2\theta}}\right.$$

$$\left.+\frac{m}{\sqrt{m^2+n^2-4a(n-a)\sin^2\theta}}\right\}\frac{(2a\sin^2\theta-n)d\theta}{n^2-4a(n-a)\sin^2\theta}$$

$$=\frac{a\rho kn(2a-n)}{n-a}\int_0^\pi\left\{\frac{l-m}{\sqrt{(l-m)^2+n^2-4a(n-a)\sin^2\theta}}\right.$$

$$\left.+\frac{m}{\sqrt{m^2+n^2-4a(n-a)\sin^2\theta}}\right\}\frac{d\theta}{n^2-4a(n-a)\sin^2\theta}$$

$$-\frac{a\rho k}{n-a}\int_0^\pi\left\{\frac{l-m}{\sqrt{(l-m)^2+n^2-4a(n-a)\sin^2\theta}}+\frac{m}{\sqrt{m^2+n^2-4a(n-a)\sin^2\theta}}\right\}d\theta,$$

$$\therefore\ Y=\frac{a\rho kn(2a-n)}{n^2(n-a)}\left[\frac{l-m}{\sqrt{(l-m)^2+n^2}}\Pi_0^\pi(-d,\ c,\ \theta)\right.$$

$$\left.+\frac{m}{\sqrt{m^2+n^2}}\Pi_0^\pi(-d,\ b,\ \theta)\right]$$

$$-\frac{a\rho k(l-m)}{(n-a)\sqrt{(l-m)^2+n^2}}E_0^\pi(c,\ \theta)-\frac{a\rho km}{(n-a)\sqrt{m^2+n^2}}E_0^\pi(b,\ \theta).$$

$$Z=2a\rho k\int_0^l\int_0^\pi\frac{2a\sin\theta\cos\theta dx d\theta}{\{(x-m)^2+n^2-4a(n-a)\sin^2\theta\}^{\frac{3}{2}}}=0.$$

F=resultant attraction $=\sqrt{X^2+Y^2+Z^2}$.

When $n=a$, the particle is on the axis of the cylinder, then

$$F=X=2\pi a\rho k\left\{\frac{1}{\sqrt{m^2+a^2}}-\frac{1}{\sqrt{(l-m)^2+a^2}}\right\}.$$

When $m=\frac{1}{2}l$, the particle is at the center of the cylinder, and $F=0$.

When $m=l$, $n=a$, $F=2\pi a\rho k\left\{\frac{1}{\sqrt{l^2+a^2}}-\frac{1}{a}\right\}$.

When $m=0$, $n=a$, $F=-2\pi a\rho k\left\{\frac{1}{\sqrt{l^2+a^2}}-\frac{1}{a}\right\}$.

When $n=2a$ the particle is on the surface of the cylinder,

$$\text{then } b^2=\frac{4a^2}{m^2+4a^2},\quad c^2=\frac{4a^2}{(l-m)^2+4a^2},\quad d=1.$$

$\therefore$ The elliptic function of the third order in Y disappears.

PROBLEMS.

42. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics and Astronomy, New Windsor College, New Windsor, Maryland.

Find the time of vibration of a particle *slightly* displaced from the center of a solid cylinder in direction of the axis, the matter of the cylinder attracting according to the laws of nature.

43. Proposed by B. F. FINKEL, A. M., Professor of Mathematics and Physics, Drury College, Springfield, Missouri.

Two weights P and Q rest on the concave side of a parabola whose axis is horizontal, and are connected by a string, length l, which passes over a smooth peg at the focus, F. [*Bowser's Analytic Mechanics, page 54.*]

DIOPHANTINE ANALYSIS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

42. Proposed by W. B. ESCOTT, 6123 Ellis Avenue, Chicago, Illinois.

In a parallelogram, sides a and b, diagonals c and d, $2a^2+2b^2=c^2+d^2$. Find all the parallelograms, not rectangles, whose sides and diagonals are rational.

Examples:	a	b	c	d
	4	7	9	7
	16	7	21	13
	8	9	13	11
	8	11	17	9

Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

By means of the sides and diagonals we can form, in each parallelogram, two different triangles, the sides of one being a, b, and c, and of the other, a, b, and d.

Take the triangle, sides a, b, and c and put $a=n$, $b=n+p$, and $c=2n\pm q$. From the relations of the sum and the difference of any two sides to the third side, we have the following conditions: $p\mp q>0$ and $p\mp q<2n$. For $p-q$, $c=2n+q$; and for $p+q$, $c=2n-q$.

The *median* upon c is $1/2\sqrt{2(a^2+b^2)-c^2}$. But as the diagonals of a parallelogram bisect each other, this median equals $\frac{1}{2}d$. Whence $d^2=2(a^2+b^2)-c^2=4n(p\mp q)+2p^2-q^2$.

Then $n=\dfrac{d^2-2p^2+q}{4(p\mp q)}$. But we have found that $2n>p\mp q$. Therefore

$\dfrac{d^2-2p^2+q^2}{2(p\mp q)}>p\mp q$. Whence $d>2p\mp q$.

Put $d=2p\mp q+t$. Then $a=n=\dfrac{(2p\mp q+t)^2-2p^2+q^2}{4(p\mp q)}$;

$b=n+p=\dfrac{(2p\mp q+t)^2+(2p\mp q)^2-2p^2}{4(p\mp q)}$;

and $c=2n\pm q=\dfrac{(2p\mp q+t)^2-(p\mp q)^2-p^2}{2(p\mp q)}$,

in which p, q, and t are any integers. p and q may also be *zero*, but only one of them in the same operation. When $p=q$ and when $q>p$, we use q only as *positive*, $[+q]$; but when $p>q$, we can use q as both *positive* and *negative*.

When numerical values, assigned to p, q, and t, render a and b or a, b, and c fractional, integral results are obtained by multiplying a, b, c, and d by the least common denominator of the fractions.

Examples:—(1). Put $p=2$, $q=1$, and $t=2$. Then, for $p+q$, $a=7/2$, $b=11/2$, $c=6$, and $d=7$; or in integers, 7, 11, 12, and 14.

(2). Put $p=3$, $q=1$, and $t=2$. Then $a=4$, $b=7$, $c=7$, and $d=9$. Also $a=4$, $b=7$, $c=9$, and $d=7$.

For $p-q$, $a=9/2$, $b=13/2$, $c=10$, $d=5$; or in integers, 9, 13, 20 and 10.

When $q=0$, or when $c=2a$, we have $a=[(2p+t)^2-2p^2]/4p$, $b=[(2p+t)^2+2p^2]/4p$, $c=[(2p+t)^2-2p^2]/2p$, and $d=2p+t$.

Examples:—(1). Put $p=1$, and $t=2$. Then $a=7/2$, $b=9/2$, $c=7$, and $d=4$; or in integers, 7, 9, 14, and 8.

(2). Put $p=t=2$. Then $a=7/2$, $b=11/2$, $c=7$, and $d=6$; or in integers, 7, 11, 14, and 12.

When $p=0$, or when $a=b$, we have $a=[(t\mp q)^2+q^2]/4q=b$,

$c=[(t\mp q)^2-q^2]/2q$, and $d=t\mp q$; or, in integral form, $a=b=(t\mp q)^2+q^2$, $c=2t(t\mp 2q)$, and $d=4q(t\mp q)$.

Examples :—(1). Put $t=q=1$. Then $a=b=5$, $c=6$, and $d=8$.

(2). Put $t=3$ and $q=1$. Then $a=b=17$, $c=30$, and $d=16$. Also $a=b=5$, $c=6$, and $d=8$.

When $q=p$, we have $a=[(3p+t)^2-p^2]/8p$, $b=[(3p+t)^2+7p^2]/8p$, $c=[(3p+t)^2-5p^2]/4p$, and $d=3p+t$.

When $t=q=p$, we have, in integral form, $a=15p$, $b=23p$, $c=22p$, and $d=32p$.

Thus we continue making general values for a, b, c, and d, under a number of other conditions; as, $t=q$; $t=p$; $t=2q=2p$, etc.

43. Proposed by M. A. GRUBER, A. M., War Department, Washington, D. C.

Find the series of integral numbers in which the sum of any two consecutive terms is the square of their difference.

I. Solution by J. H. DRUMMOND, LL. D., Portland, Maine, and the PROPOSER.

Let x and $x+m$ be two consecutive numbers. Then we have $2x+m=m^2$, and $x=m(m-1)/2$, and $x+m=m(m+1)/2$. But $m(m+1)/2$ is the sum of the terms in the series $1+2+3+4$ ....$+m$. Hence the m^{th} term of the series required is the sum of m terms of this series, and we have 1, 3, 6, 10, 15, ...$m(m-1)/2$.

II. Solution by COOPER D. SCHMITT, M. A., Professor of Mathematics, University of Tennessee, Knoxville, Tennessee; O. W. ANTHONY, M. Sc., Professor of Mathematics, New Windsor College, New Windsor, Maryland; and BENJ. F. YANNEY, A. M., Professor of Mathematics, Mount Union College, Alliance, Ohio.

By the conditions we must have $x+y=(x-y)^2$, x and y representing two consecutive terms in the series. Solving as a quadratic in x, we have $x=(2y+1)/2\pm\frac{1}{2}\sqrt{(8y+1)}$. Hence $8y+1$ must be a square.

When $y=1$, $8y+1=3^2$, $x=3$;

$y=3$, $8y+1=5^2$, $x=6$;

$y=6$, $8y+1=7^2$, $x=10$;

and the series is, 1, 3, 6, 10, 15, 21, 28, 36, 45, etc., or the system of *triangular* numbers as set forth in Pascal's Triangle.

Also solved by *A. H. HOLMES*, *E. W. MORRELL*, *H. C. WILKES*, and *G. B. M. ZERR*.

44. Proposed by A. H. HOLMES, Box 963, Brunswick, Maine.

The hypotenuse of a right-angled triangle ABC, right-angled at A, is extended equally at both extremities so that $BE=CD$. Draw AD and AE. Find integral values for all the lines in the figure thus made.

Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

Construct the figure as indicated by the problem. Then draw BF equal and parallel to AC, and draw CF, AF, EF, and DF. Then will $ABFC$ be a rectangle; and the diagonals BC and AF are equal.

It is also evident that $AE=DF$ and $AD=EF$. Whence $AEFD$ is an ob-

lique-angled parallelogram, or rhomboid, of which AE and AD are the sides, and ED and AF the diagonals.

Let $x=BE=CD$, and put $a=AD$, $b=AE$, $c=ED$, and $d=AF=BC$, taking $c>d$. Then $2x+d=c$, and $x=(c-d)/2$. If $d>c$, AE and AD fall inside of AB and AC, and the hypotenuse BC would be *contracted* instead of *extended*.

We now find integral values for a, b, c, and d. This has been done in the solution of No. 42, in this issue, and need not be reproduced here.

By this process we find integral values for all the lines except the two legs, AB and AC, of the right-angled triangle. By means of the median and the perpendicular upon BC, we readily find

$$\overline{AB}^2=d[4b^2-(c-d)^2]/4c \text{ and } \overline{AC}^2=d[4a^2-(c-d)^2]/4c.$$

Now, if these expressions can be rendered squares, without destroying the relations of a, b, c, and d, AB and AC will also be rational and integral. But I have not yet succeeded in accomplishing this. We shall now illustrate by means of a few examples.

From Diophantine problem No. 42, take the set of values, $a=4$, $b=7$, $c=9$, and $d=7$. Then $2x+7=9$; whence $x=1$. $\therefore$ $AD=4$, $AE=7$, $ED=9$, $BC=AF=7$, $BE=DC=1$, $\overline{AB}^2=112/3$, and $\overline{AC}^2=35/3$.

Take the set of values, $a=8$, $b=11$, $c=17$, and $d=9$. Then $2x+9=17$; and $x=4$. Also $\overline{AB}^2=945/17$, and $\overline{AC}^2=432/17$.

Partial solutions also received from *J. H. DRUMMOND*, *A. H. BELL*, and the *PROPOSER*.

PROBLEMS.

51. Proposed by H. C. WILKES, Skull Run, West Virginia.

The difference between the roots of two successive triangular square numbers equals the sum of two successive integral numbers, the sum of whose squares will be a square number. Demonstrate.

52. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics and Astronomy, New Windsor College, New Windsor, Maryland.

Prove that a "magic square" of nine integral elements, whose rows, columns, and diagonals have a constant sum, is only possible when this sum is a multiple of three.

AVERAGE AND PROBABILITY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

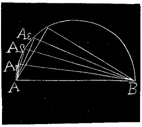
REPLY TO THE REPLIES TO MY "NOTE ON AVERAGE AND PROBABILITY."

BY ARTEMAS MARTIN, LL. D., U. S. COAST AND GEODETIC SURVEY OFFICE, WASHINGTON, D. C.

I WISH to say first that I reaffirm all that I stated on pages 370 and 371, Vol. II., No. 12, and then peoceed to consider the replies of the Repliers.

I. Professor Zerr starts out with the statement that "The problem that gives the result $\frac{1}{6}a^2$ is different from the problem that gives the result $\frac{a^2}{2\pi}$." This is superfluous information; I had clearly set forth that fact in my "Note." But the truth of the next sentence, "In the former the right angle remains fixed and does not lie on a circle as Dr. Martin states," I do not admit, and will proceed to prove its falsity.

Let $AB=a$, the given hypotenuse, which shall remain *fixed*.



Draw A_1B, A_2B, A_3B, A_4B, and so on, the sides AA_1, AA_2, AA_3, AA_4, etc., increasing uniformly from 0 towards a, the consecutive differences, AA_2-AA_1, AA_3-AA_2, AA_4-AA_3, etc., being all equal to each other, and each difference less than any assignable quantity. Thus will be had *all possible* right-angled triangles having the hypotenuse a, and, as I stated on page 371, the right angles *are* all situated on the semi-circumference whose diameter is the given hypotenuse a; but they (the right angles) are not uniformly distributed on this semicircumference because the chords AA_1, AA_2, AA_3, AA_4, etc., increase (or vary) uniformly and therefore their arcs can not increase (or vary) uniformly.

Professor Zerr continues: "The problem [the one that gives the result $\frac{1}{6}a^2$] is as follows: 'Find the average area of all triangles formed by a straight line of constant length a sliding so that its extremities constantly touch two fixed straight lines at right angles to one another'." With all due deference to Professor Zerr, I beg leave to say that I have *not* conceived the triangles to be generated in any such way, as I have clearly shown by the diagram above.

The remainder of Professor Zerr's "Note" does not require considering as it has nothing to do with the matter in hand.

II. I discard the "tail" in italics Professor Matz has appended to the problem; it is not needed to "fly the kite."

I will take up his third and fourth paragraphs. In his third paragraph he says that I, by making the number of possible right-angled triangles "proportional to the given hypotenuse," *ignore* an infinitude of right-angled tri-

angles. Now if Professor Matz can *prove* that there *are any* right-angled triangles having the hypotenuse a besides those obtained by varying one leg uniformly from 0 to a, I—would like to see the proof. How *can* there *be* any other triangles, if we have a leg for *every possible* value from 0 to a?

III. I will pass over the first and second paragraphs of the Editor's "Reply." In regard to the third paragraph I deny that any triangles *can* be interpolated, and demand proof. If one leg takes *all possible* values from 0 to a, every triangle has been included and there *can not* be any other.

IV. My solution, which I desire to reproduce here, is as follows:

Let x denote one leg of any one of the triangles, then $\sqrt{(a^2-x^2)}$ will denote the other leg. The area of this triangle is $\frac{1}{2}x\sqrt{(a^2-x^2)}$, and the true average of this is

$$\int_0^a \tfrac{1}{2}xdx\sqrt{(a^2-x^2)} \div \int_0^a dx, = \tfrac{1}{6}a^2.$$

V. I think I have considered and fully refuted every objection that has been raised against my solution.

Correction.—Vol. II., page 371, for "p. 82" read p. 282.

MISCELLANEOUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

35. Proposed by WILLIAM SYMMONDS, A. M., Professor of Mathematics and Astronomy in Pacific College, Santa Rosa, California; P. O., Sebastopol, California.

To an observer whose latitude is 40 degrees north, what is the sidereal time when Fomalhout and Antares have the same altitude; taking the Right Ascension and Declination of the former to be 22 hours, 52 minutes, —30 degrees, 12 minutes; of the latter 16 hours, 23 minutes, —26 degrees, 12 minutes?

II. Corrected solution by JOHN M. ARNOLD, Crompton, Rhode Island; and Prof. G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Let λ=latitude of observer, α, δ, α_1, δ_1 the Right Ascension and Declination of Fomalhout and Antares, respectively, β= altitude, h, h_1 the hour angles.

This event can happen only when Antares is west and Fomalhout east of the meridian.

$$\therefore \left.\begin{aligned}\sin\beta&=\sin\lambda\sin\delta+\cos\lambda\cos\lambda\cos h\\ &=\sin\lambda\sin\delta_1+\cos\lambda\cos\delta_1\cos h_1\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

$$\alpha-h=\alpha_1+h_1,\text{ or }h+h_1=\alpha-\alpha_1 \quad\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

But $\lambda=40^\circ$, $\alpha=343^\circ$, $\alpha_1=245^\circ\ 45'$. $\delta=-30^\circ\ 12'$, $\delta_1=-26^\circ\ 12'$.

$\therefore$ $662065\cos h-687337\cos h_1=39538 \ldots\ldots\ldots\ldots\ldots\ldots(3)$.

$$\cos(h+h_1)=\cos 97^\circ\ 15'=-.12620 \ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Let $\cos h=x$, $\cos h_1=y$. From (4) $y=-.12620x\pm.992005\sqrt{1-x^2}$. This in (3) gives, $748806.9294x\mp681841.7407\sqrt{1-x^2}=39538$.

$\therefore$ $x^2-.057736x=.451771$. $\therefore$ $x=.701626$ or $-.643890$.

$\therefore$ $h=45^\circ\ 26'\ 31''$ or $130^\circ\ 4'\ 57''$. The first value of h gives h_1 positive.

$\therefore$ $h=3$ hours, 1 minute, 46 seconds.

$\therefore$ sidereal time$=\alpha-h=19$ hours, 50 minutes, 14 seconds.

37. Proposed by F. M. SHIELDS, Coopwood, Mississippi.

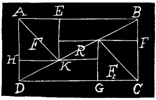
A gentleman owned and lived in the center, R, of a rectangular tract of land whose diagonal, D, was 350 rods, dividing the tract into two equal right-angled triangles, in each of which is inscribed the largest square field, F and F_1, possible; the north and south boundary lines of the two square fields being extended and joined formed a little rectangular lot, R, in the center around the residence. The difference in the area of the *entire rectangular* tract and the *sum* of the areas of the two square fields, F, F_1, is $187\frac{1}{2}$ acres. Give the dimensions and area of the entire tract, and one of the square fields, F or F_1.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science in Texarkana College, Texarkana, Arkansas-Texas.

Let $AB=a$, $AD=b$, $AH=x$. $\therefore$ $a^2+b^2=122500 \ldots\ldots\ldots\ldots\ldots\ldots(1)$.

$ab-2x^2=187\frac{1}{2}$ acres$=30000$ square rods...(2).

$ax+bx=ab \ldots\ldots\ldots\ldots\ldots\ldots(3)$,



from triangles BAD and BEK.

From (3) $x^2(a^2+2ab+b^2)=a^2b^2 \ldots\ldots\ldots(4)$.

(1) and (2) in (4) gives $62500x^2=900000000$.

$\therefore$ $x^2=14400$ square rods$=90$ acres.

$\therefore$ $x=120$ rods.

$\therefore$ $ab=58800$ square rods$=367$ acres.

$\therefore$ $a+b=490$ rods. $a-b=70$ rods. $\therefore$ $a=280$, $b=210$.

II. Solution by ISAAC L. BEVERAGE, Monterey, Virginia.

If $a=AB$ and $b=AD$, then ab=area of entire farm. Now $ab/(a+b)=AH$, since it is the side of an inscribed square of a triangle.

$\therefore$ $[ab/(a+b)]^2$=the area of F or F_1. Hence, we readily obtain,

$$ab-2[ab/(a+b)]^2=187\tfrac{1}{2}\times160 \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$\text{and }\sqrt{a^2+b^2}=350 \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Whence $a=280$ rods, and $b=210$ rods; also $ab=58800$ square rods$=367\frac{1}{2}$ acres. $\therefore ab/(a+b)=120$ rods, and $[ab/(a+b)]^2=14400$ square rods$=90$ acres.

III. Solution by A. H. BELL, Box 184, Hillsboro, Illinois.

Let 350 rods$=87\frac{1}{2}$ chains$=2a$, $DK=a-y$ and $BK=a+y$. Also, $187\frac{1}{2}$ acres$=1875$ square chains$=b^2$, and side of square$=x$; then $DG=EB=\sqrt{(a+y)^2-x^2}$, $DH=BF=\sqrt{(a-y)^2-x^2}$,

$$(\sqrt{(a+y)^2-x^2}+x)^2+(\sqrt{(a-y)^2-x^2}+x)^2=4a^2 \ldots\ldots\ldots\ldots(1).$$

Plainly, $$x\sqrt{(a+y)^2-x^2}+x\sqrt{(a-y)^2-x^2}=b^2 \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

$(2)\times 2$, and subtracted from (1), when expanded, $y^2=a^2-b^2 \ldots\ldots\ldots(3)$.

$$a+y : a-y :: \sqrt{(a+y)^2-x^2} : x. \quad \therefore x^2=(a^2-y^2)^2/2(a^2+y^2) \ldots(4).$$

Substituting values, $y=6.25$ chains, $x^2=90$ acres, $x=30$ chains, $EB=DG=40$ chains, $BF=DH=22.5$ chains, $AB=DC=70$ chains, $AD=BC=52\frac{1}{2}$ chains, $DC\times AD=367\frac{1}{2}$ acres, in the rectangle.

Also solved by *P. S. BERG*, *A. H. HOLMES*, and *B. F. YANNEY*.

PROBLEMS.

45. Proposed by EDWARD R. ROBBINS, Master in Mathematics and Physics, Lawrenceville School Lawrenceville, New Jersey.

Required several numbers each of which, divided by 10 leaves a remainder 9; by 9 leaves 8; by 8 leaves 7; by 7 leaves 6, and so on. Also find the least such number which, when divided by 28 leaves 27; by 27 leaves 26; by 26 leaves 25; by 25 leaves 24, *et cetera ad unum*.

46. Proposed by A. H. HOLMES, Box 963, Brunswick, Maine.

The base BC of the triangle ABC is $2c$, the sum of the two sides, AB and BC, is $2a$. BP is always perpendicular to AB and cuts AC in P. What is the locus of the point P?

47. Proposed by S. HART WRIGHT, A. M., Ph. D., Penn Yan, New York.

In longitude 75 degrees west of Greenwich, latitude 43 degrees, 30 minutes north on January 1, 1895, at 3 o'clock A. M., local time. What points of the ecliptic were then rising, setting and on the meridian? Any other necessary data may be taken from an ephemeris.

48. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

In case of *mischance*, with what force would the cow, weighing $w=700$ pounds, jumping over the moon, have struck Her Lunar Majesty in the face?

EDITORIALS.

Our valued contributor, Sylvester Robbins, who is visiting in Southern Ohio, made Prof. William Hoover a pleasant call a few days ago.

Professors W. W. Beman and D. E. Smith are preparing a translation of Klein's Vorträge über ausgewählte Fragen der Elementargeometrie. It will be issued during the winter by Ginn & Co.

We are grieved to record the death of our valued contributor and subscriber, Prof. H. A. Newton, of Yale University, whose death occured August 12th. In a future number of the MONTHLY will appear a biographical sketch of his life, by his colleague, Prof. A. W. Phillips.

The friends of Drury College will be pained to learn of the death of a member of its Faculty, Prof. William J. Whitney, of the Department of History, whose death, caused by typhoid fever, occurred on September 26th, at the home of his father, near Findley's Lake, New York. His broad scholarship, his accurate judgment, and the fine qualities of his character made him a great favorite among the Faculty and students of the College. Professor Whitney was a most intimate and helpful friend of Editor Finkel, and in his death we sustain a great loss.

BOOKS.

Elements of Plane and Spherical Trigonometry, A Text-book for Colleges and Schools. By Edwin S. Crawley, Ph. D., Assistant Professor of Mathematics in the University of Pennsylvania. Second edition, revised and enlarged. 8vo. Cloth, 178 pages. Price, $1.00. Published by the Author, Philadelphia, Penn.

This book contains all that is needed on the subject of Trigonometry in our best colleges. The author has omitted nothing that is necessary in studying the branches of Mathematics following Trigonometry. Such important subjects as De Moivre's Theorem, Hyperbolic Functions, Theorems relating to the escribed circles and Brocard's points are concisely treated. The book is very beautifully printed, and substantially bound in cloth. We do not hesitate to recommend this book to teachers and students desiring a good text on the subject treated. B. F. F.

Higher Mathematics. A Text-book for Classical and Engineering Colleges. Edited by Mansfield Merriman, Professor of Civil Engineering in Lehigh University, and Robert S. Woodward, Professor of Mechanics in Columbia University. Large 8vo., 576 pages. Price, $5.00. New York: John Wiley & Sons.

This volume is designed especially for the use of Junior and Senior Classes in Colleges and Technical Schools, but it is equally well adapted to the use of advanced students

and readers of Mathematics generally. The editors have called to their assistance the best mathematicians in the country, and thus given the book weight of authority never before given an American Mathematical Text-book. The book contains a concise treatment of the following subjects, not commonly found in text-books but upon which lectures are now given in our best classical and technical institutions:

Chapter I. The Solution of Equations, by Mansfield Merriman, Professor of Civil Engineering in Lehigh University; Chapter II. Determinants, by Laenas Gifford Weld, Professor of Mathematics in State University of Iowa; Chapter III. Projective Geometry, by George Bruce Halsted, Professor of Mathematics in the University of Texas; Chapter IV. Hyperbolic Functions, by James McMahon, Associate Professor of Mathematics in Cornell University; Chapter V. Harmonic Functions, by Professor William E. Byerly, Professor of Mathematics in Harvard University; Chapter VI. Functions of a Complex Variable, by Thomas S. Fiske, Adjunct Professor of Mathematics in Columbia University; Chapter VII. Differential Equations, by W. Woolsey Johnson, Professor of Mathematics in the U. S. Naval Academy; Chapter VIII. Grassmann's Space Analysis, by Edward W. Hyde, Professor of Mathematics in the University of Cincinnati; Chapter IX. Vector Analysis and Quaternions, by Alexander Macfarlane, Lecturer in Civil Engineering in Lehigh University; Chapter X. Probabilities and Theory of Errors, by Robert S. Woodward, Professor of Mechanics in Columbia University; Chapter XI. History of Modern Mathematics, by David Eugene Smith, Professor of Mathematics in Michigan State Normal School.

It is to be hoped that all classical colleges and other institutions of learning that have no provision for mathematical study in the Junior and Senior years will so arrange the course of study that the Higher Mathematics as here presented may be pursued during the last two years of college work, so that the student, during these years, may not be deprived of the rigid discipline of mind and the culture derived from its study. B. F. F.

Elementary Algebra. By H. S. Hall, M. A., and S. R. Knight, B. A. Revised and Enlarged for the use of American Schools by F. L. Sevenoak, A. M., Assistant Principal of the Academic Department, Stevens Institute of Technology. 8vo. Cloth and Leather Back. 416 pages. Price, $1.10. New York: Macmillan & Co.

Only words of commendation can be said of this book. The complete and accurate treatment of each subject, the abundant illustrations, the scientific arrangement of the subjects, go to make up all that could be desired in a good text-book. This book together with the author's Higher Algebra, makes a very exhaustive course in Algebra. B. F. F.

Euclidian Geometry. By J. A. Gillet, Professor in New York Normal College. 8vo. Cloth and Leather Back. 436 pages. New York: Henry Holt & Co.

This book, as its name implies, reverts to the purely geometric methods of Euclid. The author maintains sharply throughout the work, the distinction between the processes of pure geometry on the one hand and those of arithmetic and algebra on the other. The author says, "Euclidian Geometry bears to modern geometry very much the same relation that arithmetic bears to algebra. Its theorems are less general and it admits of positive magnitude only. For this reason its simple and rigorously logical methods can never be replaced by those of synthetic geometry, either as a factor in general education or as a foundation for advanced study." We can not agree with the Author in his last statement. It has been our experience in teaching geometry that the boy or girl, who studies geometry for the mental discipline it gives him and not merely for grades, feels better satisfied when he has demonstrated a proposition in its entirety, than he does when he has demonstrated one which he feels must be enlarged, as he advances in the study of Mathematics, to satisfy all cases. However, there is much in the book to commend it favorably to teachers. B. F. F.